

# Beyond the Thin Lens Approximation

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## Abstract

We obtain analytic formulae for the null geodesics of Friedmann-Lemaître-Robertson-Walker spacetimes with scalar perturbations in the longitudinal gauge. From these we provide a rigorous derivation of the cosmological lens equation, and obtain an expression for the magnification of a bundle of light rays without restriction to static or thin lens scenarios. We show how the usual magnification matrix naturally emerges in the appropriate limits.

**Keywords:** Cosmology: Gravitational Lensing, Gravitation

## 1. Introduction

The bending of light by a single symmetric gravitational lens in Euclidean space is shown in Fig. 1. The symmetry guarantees that the lines of sight from the observer,  $o$ , to the lens,  $l$ , and to both the lensed and unlensed images of the emitter,  $e$ , are coplanar, and can be related by the angles  $\alpha$ ,  $\beta$ , and  $\theta$ .  $D(o, l)$  and  $D(o, e)$  are the distances from the observer to the lens and emitter, respectively. We assume that the deflection angle,  $\alpha$ , is small. Then, locally about the line of sight to the emitter's image, we may approximate the two-spheres at distances  $D(o, l)$  and  $D(o, e)$  from the observer as planes, called the lens plane and source plane respectively.  $D(l, e)$  denotes the distance between these two planes. The assumption of small deflection angle also allows us to relate the lensing angles

by  $\beta = \theta - \alpha D(l, e)/D(o, e)$ . This is the simplest expression of the gravitational lens equation (Refsdal 1964).

The generalization of this equation to more complicated lens structures and non-Euclidean background spaces proceeds by a number of steps. We continue to assume that the actual path of the photon is well approximated by two segments joined at a point of deflection,  $p$ , located near the lens. A general lens is not symmetric so that the angles  $\alpha$ ,  $\beta$ , and  $\theta$  are not necessarily coplanar. To handle this, consider a set of cartesian axes with origin at the observer. Choose the  $x$ -axis to coincide with the line-of-sight to the image. Let  $\alpha^i$ , where  $i$  runs over  $\{2, 3\}$ , be the angle between the  $x$ -axis and the projection into the  $x^i x$ -plane of the line-of-sight vector from the deflection point,  $p$ , to the emitter. Similarly, let  $\beta^i$  be the angle between the projection into the  $x^i x$ -plane of the line-of-sight vector from the observer to the lens and the projection into the  $x^i x$ -plane of the line-of-sight vector from the observer to the unlensed image. Also, let  $\theta^i$  be the angle between the  $x$ -axis and the projection into the  $x^i x$ -plane of the line-of-sight vector from the observer to the lens. To allow for non-Euclidean spatial geometries  $D(o, e)$  is taken to be the angular-diameter distance in the background geometry from the observer to the intersection of the  $x$ -axis with the source plane, and  $D(l, e)$  is taken to be the angular-diameter distance in the background geometry between the deflection point,  $p$ , and the intersection of the  $x$ -axis with the source plane,  $p'$ . Then, again assuming small deflection angle,

$$\beta^i = \theta^i - \frac{D(l, e)}{D(o, e)} \alpha^i. \quad (1)$$

This is the standard cosmological gravitational lens equation (e.g. Schneider, Ehlers, and Falco 1993, Chapter 2). An important quantity associated with this formalism is the magnification matrix,  $M^i_j = \partial\beta^i/\partial\theta^j$ , which contains information on the deformation of ray bundles connecting the observer and emitter. For example, the inverse of the determinant of this matrix is the magnification of an image relative to an unlensed image.

The purpose of the current paper is to address a number of subtle issues that arise when we attempt to justify mathematically the use of the lens equation (1) for calculations in our Universe, although most workers agree that the physical basis for its use in observed lens systems is strong. First, there is the question of the correct choice of distance factors. There exists a large literature addressing this question, primarily concerned with the appropriateness of the so-called Dyer-Roeder distances (Dyer and Roeder 1972, 1973; Ehlers and Schneider 1986; Futamase and Sasaki 1989; Watanabe and Tomita 1990; Watanabe, Sasaki, and Tomita 1992; Sasaki 1993). Our work suggests that within the framework of cosmological perturbation theory, the natural distance factors to use are those of the background. Hence, the choice of distance factors is equivalent to the choice of cosmological model, in agreement with the recent results of Sasaki (1993). The issue of the most appropriate choice of cosmological model must be addressed in its own right.

The second concern in any mathematical investigation into the lens equation is the accuracy of the approximation of an actual photon path by two geodesics of the background which join at a point near the lens: the deflection point,  $p$ . On physical grounds we expect this approximation to be good for systems for which the photon-lens interaction is localized: the thin-lens approximation. One purpose of our present work is to quantify the error involved in using two geodesics of the background, rather than the actual path, in deriving the lens equation, (1).

Third, how are we to find the angles appearing in the lens equation from physical data? Generally, the  $\alpha^i$  are taken to be those calculated in Einstein-de Sitter spacetime, since the overall curvature of space should not be important near  $p$ , where the light ray interacts with the lensing object. For static, thin lenses, the deflection angle is written as a superposition of point mass deflection angles contributed by mass elements of the lens projected onto the lens plane (Schneider et al. 1993). For brevity we will term the resultant angle the “Einstein angle.” Another purpose of the present work is to derive this result from the full equations of light propagation under an appropriate set of mathematical approximations. Succinctly, the lens equation, for static, thin lenses, effectively assumes that the light path

is described by the Jacobi equation of the background spacetime subject to an impulsive wavevector deflection at the lens plane by an angle equal to the usual Einstein bending angle. We wish to quantify the level of approximation involved.

There have been two notable recent attempts to clarify the validity of the cosmological lens equation by deriving it from the optical scalar equations (Seitz, Schneider, and Ehlers 1994) and the Jacobi equation (Sasaki 1993). However, a crucial difference between these papers and the present work is that they treat the path of the light ray differently near to and far from the lens. It is precisely this assumption that we must eliminate if we hope to gain a more general lens equation able to quantify the errors implicit in equation (1).

Our approach is to investigate the cosmological lens equation as it emerges from the geodesic equation. In this our work is complementary to that of Seljak (1994) and Kaiser (1992) who have used an approach like this to investigate certain lensing questions for a perturbed Einstein-de Sitter spacetime. Where our work overlaps that of these authors we are in agreement. We will show, however, that it is possible to handle the curved Friedmann-Robertson-Walker (FRW) spacetimes by analogous calculations, though of somewhat more technical difficulty. With theorists beginning to take the idea of spatially curved models more seriously (Kamionkowski *et al.* 1993, 1994; Spergel *et al.* 1993), we feel that this extension is of more than formal importance. In this paper we make use of a method for constructing null geodesics in perturbed spacetimes introduced in Pyne and Birkinshaw (1993), hereafter PB. The method is the analog for geodesic curves of familiar perturbation techniques for differential equations. The results presented here come from applying this method to FRW spacetimes with scalar perturbations in the longitudinal gauge. Our principal results are:

- (1) analytic formulae (equations (13) and (14)) for light rays in the spacetime (5);
- (2) a general expression for the magnification undergone by a bundle of light rays capable of handling non-static, geometrically thick, lenses (equations (35), (36), and (38));
- (3) a demonstration that the usual lens equation (1) and magnification matrix are recov-

ered in the appropriate approximations.

To our knowledge, this is the first rigorous derivation of the cosmological lens equation for perturbed FRW spacetimes with spatial curvature.

The outline of this paper is as follows. In section 2 we review the perturbative geodesic expansion introduced in PB. In section 3 we apply this formalism to the problem of constructing null geodesics of FRW spacetimes with scalar perturbations. With the help of the equation of geodesic deviation for the FRW background, the solutions we find will enable us to understand the role of the Einstein angle for light propagation in these spacetimes. In section 4 we obtain an expression for the magnification of a bundle of light rays in such spacetimes without restriction to static, geometrically thin perturbations. We then discuss the emergence of the usual cosmological lens equation, (1), in the appropriate limits. In section 5 we illustrate the use of the magnification equation by solving it for a point mass embedded in an Einstein-de Sitter spacetime.

## 2. The Perturbative Geodesic Expansion

We work in geometrized units,  $G = c = 1$ . We let Greek indices  $\mu, \nu, \dots$  run over  $\{0, 1, 2, 3\}$  and Roman indices  $i, j, \dots$  run over  $\{1, 2, 3\}$ . The spacetime metric is taken to have signature  $+2$ . Our Riemann and Ricci tensor conventions are given by  $[\Delta_\alpha, \Delta_\beta] v^\mu = R^\mu{}_{\nu\alpha\beta} v^\nu$  and  $R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$ .

In PB we showed that given a null geodesic of  $g_{\mu\nu}^{(0)}$ ,  $x^{(0)\mu}(\lambda)$ , with  $\lambda$  an affine parameter, we could construct a set of four functions, which we call the separation,  $x^{(1)\mu}(\lambda)$ , transforming as a vector under infinitesimal co-ordinate change, which were sufficient to ensure that  $x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda)$  is an affinely parametrized null geodesic of  $g_{\mu\nu}^{(0)} + h_{\mu\nu}$ . In order to construct  $x^{(1)\mu}(\lambda)$  we need three quantities, two referring solely to the background metric and one referring to the perturbation. First, we need the parallel propagator (Synge, 1960) appropriate to  $x^{(0)\mu}(\lambda)$  in  $g_{\mu\nu}^{(0)}$  (in PB we called this the connector after DeFelice and Clarke (1990, Section 2.3), mostly to avoid confusion with the Jacobi propagator be-

low, but we prefer Synge's term). We remind the reader of one crucial property of the parallel propagator: if  $v^\mu(\lambda_2)$  is a vector at  $x^{(0)\mu}(\lambda_2)$  and  $v^\nu(\lambda_1)$  is its parallel translate along  $x^{(0)\mu}(\lambda)$  to the point  $x^{(0)\mu}(\lambda_1)$  then the parallel propagator  $P(\lambda_1, \lambda_2)^\mu{}_\nu$  obeys  $v^\mu(\lambda_1) = P(\lambda_1, \lambda_2)^\mu{}_\nu v^\nu(\lambda_2)$ . That is, it parallelly propagates vectors along  $x^{(0)}(\lambda)$ .

The second quantity we need, the Jacobi propagator, also refers only to the background spacetime. Introduced in PB, it is an  $8 \times 8$  dimensional matrix which serves as a Green's function for the Jacobi equation for  $g_{\mu\nu}^{(0)}$  along  $x^{(0)\mu}(\lambda)$ . Its explicit construction makes use of the parallel propagator, the curvature tensor of  $g_{\mu\nu}^{(0)}$ ,  $R^{(0)\mu}{}_{\nu\rho\sigma}$ , and the tangent vector to  $x^{(0)\mu}(\lambda)$ , the wavevector,  $k^{(0)\mu}(\lambda) = dx^{(0)\mu}(\lambda)/d\lambda$ . With eight dimensional systems the usual tensor notation can be cumbersome so we will use the matrix notation of PB. We let  $\mathcal{R}(\lambda)^\mu{}_\sigma$  denote the  $4 \times 4$  matrix  $R^{(0)\mu}{}_{\nu\rho\sigma} k^{(0)\nu} k^{(0)\rho}$  evaluated at  $x^{(0)}(\lambda)$  and write  $P(\lambda_1, \lambda)^\mu{}_\nu \mathcal{R}(\lambda)^\nu{}_\rho P(\lambda, \lambda_1)^\rho{}_\sigma$  as  $P(\lambda_1, \lambda) \mathcal{R}(\lambda) P(\lambda, \lambda_1)$ . Then the Jacobi propagator  $U(\lambda_1, \lambda_2)$  is given by

$$U(\lambda_1, \lambda_2) = \mathcal{P} \exp \left( \int_{\lambda_1}^{\lambda_2} \begin{pmatrix} 0 & 1_d \\ P(\lambda_1, \lambda) \mathcal{R}(\lambda) P(\lambda, \lambda_1) & 0 \end{pmatrix} d\lambda \right). \quad (2)$$

Here  $1_d$  is the  $4 \times 4$  identity matrix and  $\mathcal{P}$  denotes the path ordering symbol.

The final quantity we need encodes the effects of the perturbation itself. It is a vector field defined along  $x^{(0)\mu}(\lambda)$ . We denote the covariant derivative with respect to  $g_{\mu\nu}^{(0)}$  by a semi-colon. Then the perturbation vector at affine parameter  $\lambda$ ,  $f^\mu(\lambda)$ , is given by

$$f^\nu = \frac{1}{2} h_{\alpha\beta}{}^{;\nu} k^{(0)\alpha} k^{(0)\beta} - h^\nu{}_{\alpha;\beta} k^{(0)\alpha} k^{(0)\beta} \quad (3)$$

evaluated at  $x^{(0)\mu}(\lambda)$ .

It was shown in PB that the force vector, Jacobi propagator, and parallel propagator

allow us to solve for the separation,  $x^{(1)\mu}(\lambda)$ . In particular, we suppose that we desire to construct a geodesic of  $g_{\mu\nu}^{(0)} + h_{\mu\nu}$  of the form  $x^\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda)$  and that we know the appropriate boundary conditions at affine parameter  $\lambda_1$ . Then  $x^{(1)\mu}$  is given at arbitrary affine parameter,  $\lambda_2$ , by

$$\begin{aligned} \left( \frac{d}{d\lambda_2} \left[ P(\lambda_1, \lambda_2) x^{(1)}(\lambda_2) \right] \right) &= U(\lambda_2, \lambda_1) \left( \begin{array}{c} x^{(1)}(\lambda_1) \\ \left[ \frac{d}{d\lambda} \left[ P(\lambda_1, \lambda) x^{(1)}(\lambda) \right] \right]_{\lambda=\lambda_1} \end{array} \right) \\ &+ \int_{\lambda_1}^{\lambda_2} U(\lambda_2, \lambda) \left( \begin{array}{c} 0 \\ P(\lambda_1, \lambda) f(\lambda) \end{array} \right) d\lambda \end{aligned} \quad (4)$$

where the integral in this equation is taken over the background path,  $x^{(0)\mu}(\lambda)$ .

While the solution above looks complicated, in practice background spacetimes are chosen specifically for their high degree of symmetry and tractability and this often allows us to construct the needed propagators explicitly. In these cases, equation (4) reduces the work of finding null geodesics in the perturbed spacetime to a simple problem of integration. We will see below that the crucial FRW spacetimes belong to this class.

The consistency criteria for our solution are precisely those of all perturbative type geodesic calculations; extremely heuristically, the background and the constructed geodesic should not be allowed to reach regions where their spatial or temporal deviations are such that the geodesics effectively feel different gravities at equal affine parameter, either due to the perturbation or the curvature of the background itself. A simple *a priori* estimate for the domain of validity was given in PB, but in practice, the consistency may usually be checked easily after a solution is obtained. <sup>1</sup>

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<sup>1</sup> We note that the condition  $\epsilon^2 \ll \kappa$  given in PB is not, in fact, necessary. We thank Uros Seljak for pointing this out to us.

### 3. The Deflection Angle

We will now employ the techniques reviewed above to investigate the theory of gravitational lenses in cosmology. Our starting point is a choice for the metric. We choose to work with FRW spacetimes with scalar perturbations written in the longitudinal gauge,

$$d\bar{s}^2 = a^2 \left[ -(1 + 2\phi)d\eta^2 + (1 - 2\phi)\gamma^{-2} (dx^2 + dy^2 + dz^2) \right] \quad (5)$$

where  $\gamma = 1 + \kappa r^2/4$ ,  $\kappa$  being the spatial curvature parameter ( $\pm 1$  or  $0$ ) and  $r^2 = x^2 + y^2 + z^2$ . Inhomogeneities are represented by the quasi-Newtonian potential,  $\phi$ . For the order needed by us, the expansion factor,  $a(\eta)$ , is unperturbed from its Friedmann form (Jacobs, Linder, and Wagoner 1993). We choose this form for the metric for a number of reasons. The metric (5) is also used by Seitz et al. (1994) and Sasaki (1993), so that direct comparison of results is possible, and recent work by Futamase (1989) and Jacobs et al. (1993) has shown that structure of galactic scale and greater in our Universe can be well modeled by metrics of this type. While results obtained with the metric (5) are not appropriate for lensing by gravitational waves or vector perturbations, these cases are easily handled in an analogous manner.

Next we choose a particularly useful class of background light rays with which to build our perturbed solutions; radial null geodesics intersecting the observer at the spatial origin. Because the Friedmann expansion,  $a$ , plays the role of a conformal factor it is simplest to work with the null geodesics of  $ds^2$ , defined by  $d\bar{s}^2 = a^2 ds^2$ . Light rays in these two metrics coincide and their (affine) parameterizations are related by  $\bar{k}^\mu = a^{-2} k^\mu$ . With the observer located at the spatial origin, the radial null geodesics of  $ds^{(0)2}$ , (i.e. of that part of  $ds^2$  independent of  $\phi$ ), may be written  $k^{(0)0} = 1$  and  $k^{(0)i} = -\gamma e^i$ , where  $e^i$  are the direction cosines at the observer, so that  $\sum_{i=1}^3 (e^i)^2 = 1$  (McVittie 1964). Note that we have chosen our affine parameter to coincide with conformal time. This is purely for convenience. The explicit solutions for the comoving radius and for  $\gamma$  along such rays are given by



$$\begin{aligned}
r(\lambda) &= 2 \tan_{\kappa} \left( \frac{\lambda_o - \lambda}{2} \right) \\
\gamma(\lambda) &= \sec_{\kappa}^2 \left( \frac{\lambda_o - \lambda}{2} \right)
\end{aligned} \tag{6}$$

where  $\lambda_o$  is the affine parameter at the observer. The subscript  $\kappa$  on a trigonometric function denotes a set of three functions: for  $\kappa = 1$  the trigonometric function itself, for  $\kappa = -1$  the corresponding hyperbolic function, and for  $\kappa = 0$  the first term in the series expansion of the function. The paths of the rays are  $x^{(0)0} = \lambda$ ,  $x^{(0)i} = r e^i$ .

The equations of parallel transport along  $x^{(0)\mu}(\lambda)$  are easy to solve. Two vectors,  $v_2^\mu$  and  $v_1^\mu$ , at  $x^{(0)\mu}(\lambda_2)$  and  $x^{(0)\mu}(\lambda_1)$  respectively, are related by parallel translation provided that  $v_2^0 = v_1^0$  and that  $v_2^i = \gamma(\lambda_2) v_1^i / \gamma(\lambda_1)$ . We can, thus, read off the parallel propagator for our class of geodesics,

$$P(\lambda_2, \lambda_1)^\mu{}_\nu = \begin{pmatrix} 1 & 0_j \\ 0^i & \frac{\gamma(\lambda_2)}{\gamma(\lambda_1)} \delta_j^i \end{pmatrix}. \tag{7}$$

Next we will obtain the Jacobi propagator. The Riemann tensor of  $ds^{(0)2}$  is non-vanishing only when all indices are spatial, when

$$R^{(0)i}{}_{jkl} = -\kappa \left( g^{(0)i}{}_l g_{jk}^{(0)} - g^{(0)i}{}_k g_{jl}^{(0)} \right) \tag{8}$$

where  $g_{\mu\nu}^{(0)}$  is the metric  $ds^{(0)2}$ . As a result we can write  $R^{(0)\mu}{}_{\nu\rho\sigma} k^{(0)\nu} k^{(0)\rho} = -\kappa J^\mu{}_\sigma$  where

$$J^\mu{}_\sigma = \begin{pmatrix} 0 & 0_j \\ 0^i & \delta_j^i - e^i e_j \end{pmatrix}. \quad (9)$$

$J$  is idempotent,  $J^2 = J$ . This allows us easily to sum the series defining the Jacobi propagator, (2), in  $4 \times 4$  subblocks. The result is, returning fully to our matrix notation,

$$U(\lambda_2, \lambda_1) = J \otimes \begin{pmatrix} \cos_\kappa(\lambda_2 - \lambda_1) & \sin_\kappa(\lambda_2 - \lambda_1) \\ -\kappa \sin_\kappa(\lambda_2 - \lambda_1) & \cos_\kappa(\lambda_2 - \lambda_1) \end{pmatrix} \\ + (1_d - J) \otimes \begin{pmatrix} 1 & (\lambda_2 - \lambda_1) \\ 0 & 1 \end{pmatrix}. \quad (10)$$

We note that  $J$  may be interpreted as a projection operator into the space transverse to the photon direction in the comoving spatial hypersurfaces. Thus the Jacobi propagator,  $U$ , has split into a transverse rotation and a longitudinal shear.

Another crucial quantity we will need before we can construct the perturbed light rays is the force vector  $f^\mu$ . We note that the force vector may be constructed not only from (3), but also from the equivalent

$$f^\mu = \Gamma^{(1)\mu}{}_{\alpha\beta} k^{(0)\alpha} k^{(0)\beta} \quad (11)$$

where  $\Gamma^{(1)\mu}{}_{\alpha\beta}$  denotes that part of the Christoffel connection of  $ds^2$  which is linear in either the metric perturbation or its first partial derivatives. Direct calculation yields

$$f^\mu = -2 \left( \begin{matrix} k^{(0)j} \phi_{,j} \\ \phi_{,i} - k^{(0)i} k^{(0)\alpha} \phi_{,\alpha} \end{matrix} \right) \quad (12)$$

where a comma denotes an ordinary partial derivative.

It remains only to determine the appropriate initial conditions for the separation before we can use (4) to gain the null geodesics of  $ds^2$ . We will choose the perturbed geodesic and the background geodesic to coincide at the observer,  $x^{(1)\mu}(\lambda_o) = 0$ . As discussed in PB, we can not take the wavevectors of the two geodesics to coincide fully at the observer because the geodesics must each be null in their respective, different, metrics. We will choose, for convenience, with  $k^{(1)\mu} = dx^{(1)\mu}/d\lambda$ ,  $k^{(1)i}(\lambda_o) = 0$ . The constraint that  $k^{(0)\mu} + k^{(1)\mu}$  be null in  $ds^2$  at the observer then tells us that  $k^{(1)0}(\lambda_o) = -2\phi_o$  where we denote the value of  $\phi$  at the observer by  $\phi_o$ .

It is now simply a matter of assembling the necessary pieces in (4) to gain the separation at arbitrary affine parameter  $\lambda_e$ , and thus the light rays of  $ds^2$ . Some straightforward labor yields

$$x^{(1)0}(\lambda_e) = -2(\lambda_e - \lambda_o)\phi_o + 2 \int_{\lambda_o}^{\lambda_e} d\lambda (\lambda - \lambda_e) k^{(0)m} \phi_{,m}(\lambda) \quad (13)$$

and

$$\begin{aligned} x^{(1)i}(\lambda_e) = & -2k^{(0)i} \int_{\lambda_o}^{\lambda_e} d\lambda (\lambda - \lambda_e) \frac{\partial \phi}{\partial \eta}(\lambda) \\ & + 2\gamma(\lambda_e) \int_{\lambda_o}^{\lambda_e} d\lambda \sin_{\kappa}(\lambda - \lambda_e) \frac{1}{\gamma(\lambda)} (\nabla_{\perp} \phi)^i(\lambda) \end{aligned} \quad (14)$$

where  $\nabla_{\perp}^i = g^{(0)mn} (\delta_n^i - e_n e^i) \partial_m$  is the transverse gradient operator. We see that the spatial separation is naturally written as the sum of a longitudinal and a transverse term. We remind the reader that the above integrals are taken over the background geodesic  $x^{(0)\mu}(\lambda)$ . Hence if these solutions are used for geometrically thick lenses, the error terms will become important at some finite distance along the geodesic. In this case, it will in general

be necessary either to apply an iterative procedure, incorporating a number of background paths, or to appeal to statistical arguments to bound the errors. These approaches are familiar from the usual multiple lens plane theory (Schneider *et. al.* 1993; see also Seljak, 1994). A crucial difference, however, between the use of multiple background paths in our formalism and the multiple lens plane method is that formulae (13) and (14), in principal, allow the photon path to be approximated to arbitrary accuracy by successive modifications of  $x^{(0)\mu}(\lambda)$ , in contrast with the multiple lens plane theory where the continuum limit is not compatible with the assumptions underlying the theory.

It is not too difficult to check by straightforward calculation that the geodesic defined by (13) and (14) is, in fact, null. This is to be expected in light of the general theorem proved in PB that our constructed geodesic preserves its null character. We point out, however, that our appeal, in that paper, to the co-ordinate invariance of scalar quantities in order to argue for the vanishing of the term involving the partial derivative of  $g_{\mu\nu}^{(0)}$  is invalid. The theorem is, nevertheless, true.<sup>2</sup>

We can gain more understanding of (13) and (14) by considering their relation to the equation of geodesic deviation. In Appendix A we show that the Jacobi equation of  $ds^{(0)2}$  subject to an arbitrary impulsive wavevector perturbation  $\delta k^\mu$  at some affine parameter  $u$ , is solved by deviation vector  $\delta x^\mu$  with spatial components

$$\delta x^i(\lambda) = -\frac{\gamma(\lambda)}{\gamma(u)} \sin_\kappa(u - \lambda) \delta k_\perp^i(u) - \frac{\gamma(\lambda)}{\gamma(u)} (u - \lambda) \delta k_\parallel^i(u) \quad (15)$$

where  $\delta k_\perp^i = (\delta_j^i - e^i e_j) \delta k^j$  is the impulse in the transverse direction, and  $\delta k_\parallel^i = \delta k^i - \delta k_\perp^i$  is the longitudinal impulse.

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<sup>2</sup> One way to patch it up, for instance, is to note that  $g_{\mu\nu;\rho}^{(0)} = 0$  allows us to replace the offending term by a sum of two terms linear in the Christoffel symbols of the background, each of which vanishes in the chosen co-ordinate system of the second half of the proof.

A comparison of (14) and (15) leads to the interpretation of the spatial components of our solution for the separation,  $x^{(1)i}$ , as the result of a continuous sequence of impulsive perturbations

$$\delta k^i = -2 (\nabla_{\perp} \phi)^i + 2k^{(0)i} \frac{\partial \phi}{\partial \eta}. \quad (16)$$

The form of the impulse can also be gained directly from our equations by differentiating the spatial separations with respect to the affine parameter, and inserting a delta function at  $\lambda_l$  into the integrand which forces the integrand to vanish except at the lens plane. This gives

$$k^{(1)i}(\lambda_l) = -2 (\nabla_{\perp} \phi)^i(\lambda_l) + 2k^{(0)i}(\lambda_l) \frac{\partial \phi}{\partial \eta}(\lambda_l), \quad (17)$$

which is exactly the impulse found above.

At this point we need only do a little work to recover the Einstein angle from our equations. Establish a set of cartesian axes at the observer and choose the unperturbed wavevector  $k^{(0)} = (1, -\gamma, 0, 0)$ . Consider a static, localized perturbation in the  $xy$ -plane. Then the angle represented by the impulse perturbation found above (17), that is, the angle  $k^{(0)i}(\lambda_l) + k^{(1)i}(\lambda_l)$  makes with  $k^{(0)i}(\lambda_l)$ , is given by

$$\hat{\alpha}^y = \frac{-2 (\nabla_{\perp} \phi)^y}{\gamma(\lambda_l)} = -2\gamma(\lambda_l) \phi_{,y}. \quad (18)$$

The factor of  $\gamma(\lambda_l)$  is present only because our co-ordinates are scaled in an unusual way at the lens plane. If we make the co-ordinate change  $x^{i'} = x^i / \gamma(\lambda_l)$  the metric on the

lens plane becomes Minkowskian and, locally near the deflector,  $y'$  and  $z'$  serve as normal co-ordinates on the lens plane. In these co-ordinates  $\phi_{,y'}$  on the lens plane takes on the usual Newtonian form (with origin shifted from the lens, accounting for the unusual minus sign). Since our lensing angle  $-2\gamma(\lambda_l)\phi_{,y} = -2\phi_{,y'}$  the integrated impulse lensing angle for a localized perturbation is exactly the Einstein deflection angle. We emphasize that this is the first time this has been shown rigorously for the curved FRW spacetimes.

For completeness we note that the timelike component of the separation can also be analyzed by comparison to the Jacobi equation. The Jacobi equation of  $ds^{(0)2}$  for an impulse wavevector perturbation  $\delta k^\mu$  at affine parameter  $u$  results in a timelike component of the deviation vector  $\delta x^0 = -(u - \lambda)\delta k^0(u)$ . Comparison with our solution for the separation reveals that the time delay undergone by the light ray relative to the fiducial background ray may be considered to result from a sequence of impulses  $\delta k^0 = -2k^{(0)m}\phi_{,m}$ , in addition to a boundary term.

#### 4. The Magnification

We now want to examine the magnification undergone by a bundle of light rays. We define this after Schneider et al. (1993) in the following way. Suppose a source of given physical size at some redshift is observed to subtend solid angle  $d\Omega$ . An identical source observed at identical redshift placed in an FRW spacetime would subtend solid angle  $d\Omega^{(0)}$ . The magnification  $M$  is defined to be  $d\Omega/d\Omega^{(0)}$ .

To gain the magnification we will construct an infinitesimal bundle of light rays in  $d\bar{s}^2$  which emanate from a source and converge at an observer, located at the spatial origin of co-ordinates, by varying the direction cosines of the background ray,  $e^i$ , in equations (13) and (14). We will determine the solid angle,  $d\Omega$ , subtended by the rays in the rest frame of an observer with four-velocity  $u_o^\mu = (1/a_o)(1 - \phi_o, v_o^i)$ . We will then ask what local area transverse to its direction of propagation the bundle sweeps out in a frame with four velocity  $u_e^\mu = (1/a_e)(1 - \phi_e, v_e^i)$ ,  $dA$ , at a given redshift,  $z$ . We regard the peculiar

velocities  $v_o^i$  and  $v_e^i$  as first order quantities so that  $u_o^\mu$  and  $u_e^\mu$  are properly normalized to first order in  $d\bar{s}^2$ . We suppose a source with four-velocity  $u_e^\mu$  intersects our bundle, that its redshift is  $z$ , that its physical size is  $dA$ , and that its shape such that it exactly fills the beam; that is, our bundle is the light of a physical source. In this way we will gain a relationship between  $d\Omega$  and the redshift and proper size of the source and the four-velocity of the observer. A similar relationship is easy to derive for an identical source in the background spacetime. Comparison of the two expressions will yield  $M$ . Figure 2 illustrates the constructions of this section.

We begin by choosing a set of null geodesics of the background with which to construct our congruence. We use the two-parameter family of curves given by  $x^{(0)\mu} = (\lambda, re^i)$  with  $r$  as in (6),  $e^i = (1, d \sin \theta, d \cos \theta)$ ,  $d \in (0, \epsilon)$  with  $\epsilon$  infinitesimal, and  $\theta \in (0, 2\pi]$ . We work to first order in  $\epsilon$ . To this order, this set of rays defines a null congruence of  $ds^{(0)2}$ . To each of the rays of this congruence is associated a null ray of  $ds^2$ , and hence of  $d\bar{s}^2$ , by (13) and (14) above,  $x^\mu(\lambda; d, \theta)$ . In the rest space of the observer, these rays define a cone. To see this, we note that the conical shape is clear for a comoving observer. For an observer with some peculiar velocity the assertion then follows from a result of Sachs (1961) on the geometry of null rays.

Next we will determine the two-dimensional projected area of our bundle at a given redshift. In Appendix B it is proven that, to first order in the perturbation,  $w^\mu = x^\mu(d = \epsilon, \theta) - x^\mu(d = 0)$  is a one-parameter family of geodesic deviation vectors of  $d\bar{s}^2$  along the central geodesic of the bundle,  $x^\mu(d = 0)$ . From this point on, any wavevector or path pertaining either to the perturbed or unperturbed spacetimes not written with an explicit  $d$ -argument is intended to refer to the appropriate central geodesic,  $d = 0$ . Taylor expansion writes

$$w^\mu = \left( \frac{\partial x^{(0)\mu}}{\partial e^j} + \frac{\partial x^{(1)\mu}}{\partial e^j} \right)_{e^i=(1,0,0)} \left( \epsilon \sin \theta \delta_2^j + \epsilon \cos \theta \delta_3^j \right), \quad (19)$$

which we can write as

$$w^\mu = w_{(2)}^\mu \epsilon \sin \theta + w_{(3)}^\mu \epsilon \cos \theta, \quad (20)$$

with

$$w_{(j)}^\mu = \frac{\partial x^\mu}{\partial e^j} \Big|_{e^i=(1,0,0)} \quad (21)$$

for  $j = 2, 3$ . It will also be useful to define, with  $j = 2, 3$ ,

$$w_{(j)}^{(0)\mu} = \frac{\partial x^{(0)\mu}}{\partial e^j} \Big|_{e^i=(1,0,0)} = r \delta_j^\mu, \quad (22)$$

and

$$w_{(j)}^{(1)\mu} = \frac{\partial x^{(1)\mu}}{\partial e^j} \Big|_{e^i=(1,0,0)}, \quad (23)$$

so that  $w_{(j)}^\mu = w_{(j)}^{(0)\mu} + w_{(j)}^{(1)\mu}$ .

Suppose now that the central geodesic of our bundle intersects the worldline of our hypothetical source at the point  $e$ . The projection of  $w^\mu$  into the two-dimensional subspace orthogonal to both  $u_e^\mu$  and to the projection of  $\bar{k}^\mu$  into this subspace, i.e. to the direction of photon propagation, defines an ellipse. We can determine the characteristics of this ellipse explicitly. The relevant projection operator is given by (Kristian and Sachs 1965)



$$H^\mu{}_\nu = \delta^\mu_\nu - \frac{\bar{k}^\mu \bar{k}_\nu}{(u_e \cdot \bar{k})^2} - \frac{u_e^\mu \bar{k}_\nu}{u_e \cdot \bar{k}} - \frac{\bar{k}^\mu u_{e\nu}}{u_e \cdot \bar{k}} \quad (24)$$

where we denote the  $d\bar{s}^2$  inner product by a dot, e.g.  $u_e \cdot \bar{k} = u_e^\alpha \bar{g}_{\alpha\beta} \bar{k}^\beta$ . We will use  $\perp$  to denote the result of acting on a given vector with  $H^\mu{}_\nu$ , e.g.  $w_\perp^\mu = H^\mu{}_\nu w^\nu$ .

Extremizing  $w_\perp \cdot w_\perp$  with respect to  $\theta$  we find the major and minor axes of the ellipse occur for

$$\theta = \frac{1}{2} \tan^{-1} \left( \frac{-2w_{\perp(2)} \cdot w_{\perp(3)}}{[w_{\perp(2)} \cdot w_{\perp(2)} - w_{\perp(3)} \cdot w_{\perp(3)}]} \right) + \frac{n\pi}{2} \quad (25)$$

with  $n = 0, 1$ . In (25) the inner products are evaluated at  $e$ . Denoting the argument of the inverse tangent by  $\chi$ , the squared lengths of the two axes are given by, for  $\theta = (1/2) \tan^{-1} \chi$

$$w_\perp \cdot w_\perp = \frac{\epsilon^2}{2\sqrt{1+\chi^2}} \left[ (w_{\perp(2)} \cdot w_{\perp(2)} + w_{\perp(3)} \cdot w_{\perp(3)}) \sqrt{1+\chi^2} - (w_{\perp(2)} \cdot w_{\perp(2)} - w_{\perp(3)} \cdot w_{\perp(3)} - 2\chi w_{\perp(2)} \cdot w_{\perp(3)}) \right] \quad (26)$$

and for  $\theta = -\pi/2 + (1/2) \tan^{-1} \chi$

$$w_\perp \cdot w_\perp = \frac{\epsilon^2}{2\sqrt{1+\chi^2}} \left[ (w_{\perp(2)} \cdot w_{\perp(2)} + w_{\perp(3)} \cdot w_{\perp(3)}) \sqrt{1+\chi^2} + (w_{\perp(2)} \cdot w_{\perp(2)} - w_{\perp(3)} \cdot w_{\perp(3)} - 2\chi w_{\perp(2)} \cdot w_{\perp(3)}) \right] \quad (27)$$

which lead to the area of the ellipse,  $dA$ , being

$$dA = \epsilon^2 \pi \left[ \left( w_{\perp(2)} \cdot w_{\perp(2)} \right) \left( w_{\perp(3)} \cdot w_{\perp(3)} \right) - \left( w_{\perp(2)} \cdot w_{\perp(3)} \right)^2 \right]^{1/2}. \quad (28)$$

At this point the underlying structure of our calculation is becoming clear. We are involved in a realization of standard ideas in the geometry of linear maps of the plane: by Jacobi propagation followed by projection we have linearly mapped the two plane spanned by  $w_{(j)}^\mu$ ,  $j = 2, 3$ , to the transverse two-plane at the source. Much of this section is easier to understand with this in mind.

We continue by breaking the projection operator up into zeroth and first order parts in the perturbation  $H^\mu{}_\nu = H^{(0)\mu}{}_\nu + H^{(1)\mu}{}_\nu$ . It is then possible to show that to first order  $H^\mu{}_\nu w^\nu H_{\mu\alpha} w^\alpha = H^{(0)\mu}{}_\nu w^\nu H_{\mu\alpha}^{(0)} w^\alpha$ . This relationship has a simple geometrical interpretation. By a theorem of Sachs (1961), the four-velocity of the emitter has no effect on the size and shape of our ellipse so we take the emitter to have  $v_e^i = 0$ . In this case  $H$  and  $H^{(0)}$  project onto two planes inclined with respect to each other by a small angle, the lens angle at the emitter. But the projected areas in two such planes differ only by a factor of  $\cos \alpha$ ,  $\alpha$  the angle of inclination. Since  $\alpha$  is first order, the areas are equal to first order. Therefore in the expression for the area, (28) above, we are able to replace all  $H$ -projected quantities by  $H^{(0)}$ -projected quantities.

Next we use the explicit form of the projector to find  $H^{(0)\mu}{}_\nu w^\nu = (0, 0, w^y, w^z)$ . Substituting back into (28) then yields

$$\begin{aligned} dA &= \epsilon^2 \pi \frac{a_e^2}{\gamma_e^2} (1 - 2\phi_e) \text{Det} \begin{pmatrix} w_{(2)}^y & w_{(3)}^y \\ w_{(2)}^z & w_{(3)}^z \end{pmatrix} \\ &= \epsilon^2 \pi \frac{a_e^2 r_e^2}{\gamma_e^2} (1 - 2\phi_e) \text{Det} M^i{}_j \end{aligned} \quad (29)$$

where the magnification matrix,  $M^i{}_j$  is given by

$$M^i_j = 1_d + \frac{1}{r} w_{(j)}^{(1)i}, \quad (30)$$

with  $i, j \in \{2, 3\}$  and  $1_d$  the  $2 \times 2$  identity matrix. In (29) we have written  $r_e$  for  $r(\lambda_e)$  and  $\gamma_e$  for  $\gamma(\lambda_e)$ .  $M^i_j$  is a function of the affine parameter and in both the above equations is evaluated at  $\lambda = \lambda_e$ , corresponding to the point  $e$  along the central perturbed geodesic. We may recognize the first of the equalities in (29) as the transformation of area law for infinitesimal linear mappings of the plane once we recognize the factors in front of the determinant (aside from the factor of  $\pi$ ) as corresponding to the metric factors in the induced area form for our image two-plane. Equivalently, this is the induced area two-form on the  $yz$ -plane acting on  $H^{(0)}w_{(2)}$  and  $H^{(0)}w_{(3)}$  (again up to the factor of  $\pi$ ).

The above expression (30) for the magnification matrix is one of the key results of this section. We recall from (23) that for  $i, j = 2, 3$ ,  $w_{(j)}^{(1)i}$  is precisely the variation of the transverse separation with the transverse direction cosine. The factor of  $r^{-1}$  in (30) turns the transverse separation into the transverse angular deflection. Thus, we are beginning to see the usual structure of the magnification matrix emerge. We notice, however, from (14), that while the time variation of the potential does not contribute to the transverse deflection of a single ray it will contribute to the gradient in (23) (this will become clear below), so that non-static potentials contribute to the magnification matrix. We note that (30) may be used for vector and tensor perturbations in addition to the scalar ones considered here, after a change in the solution for the separation, hence  $w_{(j)}^{(1)i}$ , which arises from a change in the specific form of the force vector,  $f^\mu$ .

We can use (29) to find  $d\Omega$ , the solid angle in the rest frame of the observer defined by our bundle. To this end, let  $\lambda_e = \lambda_o + \Delta\lambda$ ,  $\Delta\lambda$  small. We will use (29) to express the projected area of our bundle at  $\lambda_e$  to order  $(\Delta\lambda)^2$ . Noting that  $\gamma \sin_\kappa(\lambda_o - \lambda) = r$ , we have

$$dA(\lambda_o + \Delta\lambda) = \pi\epsilon^2 a_o^2 (\Delta\lambda)^2 (1 - 2\phi_o) \quad (31)$$

The proper spatial distance in the rest frame of  $u_o^\mu$  in the metric  $d\bar{s}^2$  corresponding to an affine distance  $\Delta\omega$  along  $\bar{k}^{(0)\mu}$ ,  $\Delta L$ , is given by (Ellis, 1971)  $\Delta L = |\Delta\omega| (u \cdot \bar{k})_o$ . Since  $\Delta\omega = a_o^2 \Delta\lambda$  we can write the projected area of the bundle a unit spatial distance away from the observer as

$$\begin{aligned} d\Omega &= dA(\Delta L = 1) \\ &= \pi\epsilon^2 a_o^{-2} \frac{(1 - 2\phi_o)}{(u_o \cdot \bar{k}_o)^2} \\ &= \pi\epsilon^2 (1 + 2v_o^i k_i^{(0)}(\lambda_o)) \end{aligned} \quad (32)$$

where in the last equality we have used  $u_o^\mu = (1/a_o)(1 - \phi_o, v_o^i)$  and  $\bar{k}_o^\mu = a_o^{-2} k_o^{(0)\mu} + a_o^{-2} k_o^{(1)\mu} = a_o^{-2}(1 - 2\phi_o, -1, 0, 0)$ . In fact (32) is simply the Lorentz transformation law for solid angle to linear order in  $v_o^i$ . Were the observer comoving, the solid angle would be given by  $\pi\epsilon^2$ . The expression (32) comes from transforming to a frame in which the observer moves with velocity  $v_o^i$ .

We can use this expression to replace  $\epsilon^2\pi$  in (29), yielding

$$dA = d\Omega (1 - 2v_o^i k_i^{(0)}(\lambda_o)) \frac{a_e^2 r_e^2}{\gamma_e^2} (1 - 2\phi_e) \text{Det} M^i_j(\lambda_e) \quad (33)$$

This is the desired expression relating the proper area of our emitter and the solid angle that it is observed to subtend given its affine distance,  $\lambda_e$  (which we will soon trade in for

its redshift). The equivalent expression for a congruence of the background spacetime can be gained from (33) by taking the perturbed quantities to vanish. This gives

$$dA^{(0)} = d\Omega^{(0)} \frac{a_q^2 r_q^2}{\gamma_q^2} \quad (34)$$

where  $q$  is some point along the central background geodesic,  $x^{(0)\mu} = (\lambda, r, 0, 0)$  (see Fig. 2).

To proceed we must ensure that the points  $e$  and  $q$  correspond to the same numerical source redshifts in their respective spacetimes. Let the affine parameter corresponding to the point  $q$  be given by  $\lambda_q$ . We need to impose  $1 + z^{(0)}(\lambda_q) = 1 + z(\lambda_e)$ , where  $z^{(0)}$  refers to the redshift to the source in the background along the central background geodesic and  $z$  refers to the redshift of the source in the perturbed spacetime along the central perturbed geodesic. The redshift in the perturbed spacetime is given by  $1 + z = (u_e \cdot \bar{k}) / (u_o \cdot \bar{k})(\lambda_o)$ .  $1 + z^{(0)}$  may be found from this expression simply by forcing the perturbed quantities to vanish. If we write  $\lambda_e = \lambda_q + \delta\lambda_q$  with  $\delta\lambda_q$  a first order function of  $\lambda_q$ , the equal redshift constraint is solved by

$$\delta\lambda_q = \frac{a_q}{\dot{a}_q} \left( v_o^i k_i^{(0)}(\lambda_o) - v_e^i k_i^{(0)}(\lambda_q) + \left[ \phi - \dot{a} a^{-1} x^{(1)0} + k^{(1)0} \right]_o^q \right). \quad (35)$$

Here a subscript  $q$  denotes evaluation at the point  $q$ , an overdot denotes an exact conformal time derivative along the comoving worldlines,  $[f]_o^q = f_q - f_o$ , and the separation and its affine derivative are solved for along  $k^{(0)\mu}$ , which intersects both  $o$  and  $q$  by construction. We have also noted that, to first order,  $\phi_e = \phi_q$ . We have not used the analogous formula for the emitter's peculiar velocity simply to emphasize that the emitter can be moving, in principle, under many different types of non-gravitational forces and so the notion of a

smoothly varying peculiar velocity field may be inappropriate.<sup>3</sup>

Recalling the definition of the magnification, we now set  $dA^{(0)}$  in (34) equal to  $dA$  in (33) and expand  $a_e$ ,  $r_e$ , and  $\gamma_e$  about their values at  $q$ . We also use the equivalence of  $M^i_j(\lambda_e)$  and  $M^i_j(\lambda_q)$  to first order. The end result is the following formula for the magnification of a source observed at redshift  $z^{(0)}(\lambda_q)$ ;

$$M = \frac{1}{(\text{Det} M^i_j)(\lambda_q)} \left[ 1 - 2\phi_o + 2v_e^i k_i^{(0)}(\lambda_q) - 2k^{(1)0}(\lambda_q) + 2\cot_\kappa(\lambda_o - \lambda_q) \delta\lambda_q + \kappa \sin_\kappa(\lambda_o - \lambda_q) e_i x^{(1)i}(\lambda_q) \right]. \quad (36)$$

We emphasize that the source redshift in the actual perturbed spacetime is given numerically by  $z^{(0)}(\lambda_q)$ . We have chosen to express  $\lambda_e$  in terms of  $\lambda_q$  rather than the other way round because this choice makes it simple to take the physical redshift as the independent variable in the magnification formula, (36), rather than  $\lambda_q$ .

An explicit formula for the magnification matrix may be found directly from (30), (23) and (14). We need to take a partial derivative of (14) with respect to  $e^i$ ,  $i = 2, 3$ , while keeping  $\lambda$  fixed. The only subtlety arises from the implicit dependence of  $\phi$  and its spacetime derivatives on  $e^i$  which arises because of the need to evaluate these quantities on a particular unperturbed geodesic. This means

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<sup>3</sup> We have, however, considered  $v_e$  and  $v_o$  to contribute, in a numerical sense, only at first order. For instance, the source and emitter four-velocities are properly normalized only if  $v_{o(e)}$  are considered to be first order quantities. It is not difficult to relax this assumption.

$$\begin{aligned}
\frac{\partial \phi}{\partial e^j} &= \frac{\partial \phi}{\partial x^\alpha} \frac{\partial x^{(0)\alpha}}{\partial e^j} \\
&= \frac{\partial \phi}{\partial x^\alpha} w^{(0)\alpha}_{(j)} \\
&= r \frac{\partial \phi}{\partial x^j}.
\end{aligned} \tag{37}$$

In principle  $\gamma$ , and hence  $g_{\mu\nu}^{(0)}$ , have a similar implicit dependence on  $e^i$ , but in fact this dependence is vanishing. This is easy to understand once it is realized that varying  $e^i$  for  $i = 2, 3$  with fixed  $\lambda$  amounts to variation tangent to the two-sphere at fixed co-ordinate radius  $r(\lambda)$ . Thus  $r$ , and so  $\gamma$  and  $g_{\mu\nu}^{(0)}$ , are stationary. Keeping the above in mind, we find an explicit formula for the magnification matrix without restriction to either static or thin perturbations (i.e. lenses). With  $i, j = 2, 3$ ,

$$\begin{aligned}
M^i_j &= \delta_j^i + \frac{2}{\sin_\kappa(\lambda_o - \lambda_q)} \delta_j^i \int_{\lambda_o}^{\lambda_q} d\lambda (\lambda - \lambda_q) \frac{\partial \phi}{\partial \eta}(\lambda) \\
&\quad - \frac{2}{\sin_\kappa(\lambda_o - \lambda_q)} \int_{\lambda_o}^{\lambda_q} d\lambda \sin_\kappa(\lambda - \lambda_q) \gamma(\lambda) \\
&\quad \times \left[ \delta_j^i \frac{\partial \phi}{\partial x}(\lambda) - r(\lambda) \gamma^{-2}(\lambda) g^{(0)ik}(\lambda) \phi_{,kj}(\lambda) \right].
\end{aligned} \tag{38}$$

While (38) is computationally quite useful, its relationship to the usual form for the magnification matrix is easier to see after rewriting. Introducing the angular diameter distance of  $d\bar{s}^{(0)}$ ,  $\bar{D}(\lambda_1, \lambda_2) = a(\lambda_2) \sin_\kappa(\lambda_1 - \lambda_2)$ , (see Appendix A), and using the two-dimensional projected angle of section 1 above,

$$\hat{\alpha}^i = -2 \frac{(\nabla_\perp \phi)^i}{\gamma}, \tag{39}$$

we can write

$$M^i_j = \delta^i_j + 2 \frac{a(\lambda_q)}{\bar{D}(\lambda_o, \lambda_q)} \delta^i_j \int_{\lambda_o}^{\lambda_q} d\lambda (\lambda - \lambda_q) \frac{\partial \phi}{\partial \eta}(\lambda) - \frac{1}{\bar{D}(\lambda_o, \lambda_q)} \int_{\lambda_o}^{\lambda_q} d\lambda \bar{D}(\lambda, \lambda_q) \frac{\partial \hat{\alpha}^i}{\partial e^j}(\lambda). \quad (40)$$

Finally, we note that to zeroth order  $\partial/\partial e^i = \partial/\partial \theta^i$  with  $\theta^i$  the vectorial angle of (1). This allows us to make the replacement

$$\frac{\partial \hat{\alpha}^i}{\partial e^j} \rightarrow \frac{\partial \hat{\alpha}^i}{\partial \theta^j}, \quad (41)$$

in (40). The result is the most elegant form of our equation. It is also the easiest form in which to recover the thin lens limit. For this, suppose the potential is that appropriate to a static, geometrically thin lens. Then the first term in (40) vanishes and we can approximate the angular diameter factor as constant over the region for which the potential is important. Finally, we replace  $\lambda_q$  with  $\lambda_e$  since, to first order, they are equivalent in  $M^i_j$ . The result is

$$M^i_j \approx \delta^i_j - \frac{\bar{D}(\lambda_l, \lambda_e)}{\bar{D}(\lambda_o, \lambda_e)} \frac{\partial}{\partial \theta^j} \int_{\lambda_o}^{\lambda_e} d\lambda \hat{\alpha}^i. \quad (42)$$

We have already seen that the integral of  $\hat{\alpha}^i$  over the background path produces the Einstein deflection angle. As a result we conclude that in this limit our equation has reproduced the usual magnification matrix defined by the  $\theta$ -gradient of equation (1).



## 5. The Thin Lens in Einstein-de Sitter Space

As an explicit illustration of the ideas above we will solve for the magnification matrix appropriate to a point mass perturbation of Einstein-de Sitter spacetime. We will find that the usual expression of the magnification matrix is correct provided that the impact parameter is much the smallest lengthscale in the problem (save, of course, for the Schwarzschild radius of the point mass). Certainly the calculation below is the hard way to produce this result. Nevertheless, for the first time the standard result is obtained along with correction terms arising from the time variation of the perturbation and the difference between the actual path and its piecewise geodesic approximation.

Our starting point is the potential approximation to general relativity (Martinez-Gonzalez, Sanz, and Silk 1990), which writes the perturbation to the flat FRW metric in (5) appropriate to a comoving point particle of mass  $m$  located at  $(x_l, y_l, z_l)$  on the spatial hypersurfaces of constant conformal time as

$$\phi(\eta, \vec{x}) = \frac{-m}{a(\eta)\sqrt{(x-x_l)^2 + (y-y_l)^2 + (z-z_l)^2}}. \quad (43)$$

It is worth noting that this perturbation is not static. Its time dependence, however, is simply that of the background spacetime, that is, it is set by the cosmology. We certainly do not expect time variation of this magnitude to affect photons streaming past the lens. We will see this prejudice borne out in the calculations below. Nevertheless, this emphasizes that more complicated details will emerge in the rigorous picture of lensing than in the usual models.

We will suppose that, other than the point mass, the observer, and the lensed source, the spacetime is filled with dust so that we can take  $a = a_o\eta^2/\eta_o^2$  with the subscript  $o$  denoting evaluation at the observer and  $a_o$  constant with the dimension of length (McVittie 1964). It is useful to keep in mind that with our conventions the only dimensionful numbers

in this problem are  $a_o$  and  $m$  and both have dimensions of length.

To avoid unnecessary complications we will take both the source and the emitter to be comoving, as would be true, for instance, if both were far from the point mass. We suppose the observer to be at the spatial origin of co-ordinates with the source image toward the positive  $x$  direction. The light from the source does not travel exactly down the  $x$ -axis because it is bent by the action of the lens. We could calculate the bending by constructing the path using (13) and (14) above. Instead we will use (38) to gain the magnification matrix directly. The background path appropriate to the situation is given by  $x^{(0)\mu}(\lambda) = (\lambda, \lambda_o - \lambda, 0, 0)$  with  $\lambda_o$  the affine parameter at the observer. We denote by  $\lambda_l$  that value of the affine parameter at which  $x^{(0)\mu}$  intersects the plane  $x = x_l$ , i.e.  $x_l = \lambda_o - \lambda_l$ . For the usual lensing scenarios, this point of intersection is very nearly the point of closest approach of the photon to the lens.

Given our geometry, the physical linear size of the impact parameter,  $b$ , is given by

$$b = a(\lambda_l) \sqrt{y_l^2 + z_l^2} \quad (44)$$

so that the angle between the lens and the image of the emitter,  $\theta$ , is given by

$$\theta = \frac{a(\lambda_l) \sqrt{y_l^2 + z_l^2}}{\bar{D}(\lambda_o, \lambda_l)} \quad (45)$$

It will be convenient to define an unbarred angular-diameter distance symbol,  $D_{12}$ , by  $D_{12} = \bar{D}(\lambda_1, \lambda_2) / a(\lambda_2) = (\lambda_1 - \lambda_2)$  so that

$$\theta = \frac{\sqrt{y_l^2 + z_l^2}}{D_{ol}} \quad (46)$$

Also, in terms of the vectorial decomposition of  $\theta$  introduced in section 1, we have  $x_l^i = \theta^i D_{ol}$ ,  $i = 2, 3$ . The physical situation described by the above geometry is illustrated in Fig. 3.

We start the calculation by noting that for an Einstein-de Sitter background we may combine the second and third terms in (38) to give

$$M^i_j = \delta_j^i + \frac{2}{D_{oq}} \delta_j^i \int_{\lambda_o}^{\lambda_q} d\lambda (\lambda - \lambda_q) \frac{d\phi}{d\lambda}(\lambda) + \frac{2}{D_{oq}} \int_{\lambda_o}^{\lambda_q} d\lambda \left( (\lambda_o + \lambda_q) \lambda - \lambda_o \lambda_q - \lambda^2 \right) g^{(0)ik} \frac{\partial \phi}{\partial x^k \partial x^j}(\lambda) \quad (47)$$

The details of our model lens then transform this into

$$M^i_j = (1 - 2\phi_o) \delta_j^i + \frac{2\eta_o^2 m}{a_o D_{oq}} [I_{2,1} - I_{0,3} + (\lambda_o + \lambda_q) I_{1,3} - \lambda_o \lambda_q I_{2,3}] \delta_j^i + \frac{6\eta_o^2 m}{a_o D_{oq}} [I_{0,5} - (\lambda_o + \lambda_q) I_{1,5} + \lambda_o \lambda_q I_{2,5}] x_l^i x_l^j \quad (48)$$

where we have put

$$I_{k,n} = \int_{\lambda_o}^{\lambda_q} d\lambda \frac{1}{\lambda^k \left[ (\lambda_l - \lambda)^2 + y_l^2 + z_l^2 \right]^{n/2}} \quad (49)$$

As all of these integrals may be explicitly performed, a general (but complicated and uninteresting) expression for the magnification matrix may be written down. For illustration, we will consider the situation that the impact parameter is much smaller than all the other physical sizes in the problem, so that the integrals simplify to

$$\begin{aligned}
I_{2,5} &= -\frac{4}{3\lambda_l^2\theta^4 D_{ol}^4} - \frac{5}{\lambda_l^6}\Delta \\
I_{1,5} &= -\frac{4}{3\lambda_l\theta^4 D_{ol}^4} + \frac{1}{\lambda_l^5}\Delta \\
I_{0,5} &= -\frac{4}{3\theta^4 D_{ol}^4} \\
I_{2,3} &= -\frac{2}{\lambda_l^2\theta^2 D_{ol}^2} + \frac{3}{\lambda_l^4}\Delta \\
I_{1,3} &= -\frac{2}{\lambda_l\theta^2 D_{ol}^2} + \frac{1}{\lambda_l^3}\Delta \\
I_{0,3} &= -\frac{2}{\theta^2 D_{ol}^2} \\
I_{2,1} &= -\frac{D_{lq}}{\lambda_l^2\lambda_q} + \frac{D_{ol}}{\lambda_l^2\lambda_o} + \frac{1}{\lambda_l^2}\Delta
\end{aligned} \tag{50}$$

where

$$\Delta = \left[ -\sinh^{-1} \left( \frac{\lambda_l (\lambda_l - \lambda)}{\lambda\theta D_{ol}} \right) \right]_{\lambda_o}^{\lambda_q} \tag{51}$$

With somewhat more algebra it can be shown that  $I_{2,1}$  and all the terms proportional to  $\Delta$  contribute negligibly to the magnification matrix under the assumption of small impact parameter. The remaining terms combine to produce

$$M^i{}_j = -\frac{4\eta_o^2 m D_{lq}}{a_o \lambda_l^2 D_{ol} D_{oq} \theta^2} \delta_j^i + \frac{8\eta_o^2 m D_{lq} \theta^i \theta^j}{a_o \lambda_l^2 D_{ol} D_{oq} \theta^4} \tag{52}$$

which, because  $q$  and  $e$  are equivalent to the needed order, is also expressible as

$$M_j^i = -\frac{4m\bar{D}_{le}}{\bar{D}_{ol}\bar{D}_{oe}} \left( \frac{1}{\theta^2} \delta_j^i - 2 \frac{\theta^i \theta^j}{\theta^4} \right) \quad (53)$$

This agrees exactly with the usual magnification matrix for the point mass lens (Schneider *et al.* 1993, Chapter 2).

## 6. Summary

We have presented formulae (13), (14) for the null geodesics intersecting an observer's worldline in an important class of perturbed spacetimes, FRW backgrounds with scalar perturbations, in the longitudinal gauge. We have used these equations to obtain a general formula (36) for the magnification of ray bundles in these spacetimes. With this, we can show for the first time how the usual lens equation (1) and magnification matrix are recovered in the curved FRW spacetimes without dividing light paths into near and far lens regions. To illustrate our formulae we have calculated the magnification matrix appropriate to a point deflector in an Einstein-de Sitter spacetime. We are able to show how the usual formula emerges along with (in this case negligible) correction terms. The techniques used in this paper are easily applicable to FRW spacetimes with vector or tensor perturbations.

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## Appendix A: The Jacobi Equation of FRW Spacetimes

We consider the equation of geodesic deviation in the spacetimes  $ds^{(0)2}$ , related by conformal transformation to the background FRW spacetimes. The equation of geodesic deviation along  $k^{(0)\mu}$  is

$$\frac{D^2}{d\lambda^2}\delta x^\mu = R^{(0)\mu}{}_{\alpha\beta\gamma}k^{(0)\alpha}k^{(0)\beta}\delta x^\gamma. \quad (\text{A1})$$

Here  $D/d\lambda$  is the covariant derivative along  $k^{(0)\mu}$ . We will take  $k^{(0)\mu}$  to be the wavevector of a radial null geodesic parameterized as in section 3 above. Changing variables via  $\delta x^\mu(\lambda) = P(\lambda, a)^\mu{}_\alpha v^\alpha(\lambda)$ ,  $a$  an arbitrary affine parameter value along  $k^{(0)\mu}$ , allows us to write (A1) as

$$\frac{d^2}{d\lambda^2}v^\mu(\lambda) = -\kappa J^\mu{}_\alpha v^\alpha(\lambda). \quad (\text{A2})$$

We want to solve this equation subject to the initial conditions  $\delta x^\mu(u) = 0$  and  $d\delta x^\mu/d\lambda(u) = \delta k^\mu(u)$ . These initial conditions describe some impulse which jolts the geodesic at some affine parameter distance  $u$ .

(A2) conveniently decomposes into three separate equations, one for each of the timelike component, and transverse and longitudinal projections of the spatial components. To see this, multiply (A2) by unity in the form of  $(\delta^\alpha_\beta - J^\alpha_\beta) + J^\alpha_\beta$ . The result is

$$\begin{aligned}
\frac{d^2}{d\lambda^2}v^0 &= 0 \\
\frac{d^2}{d\lambda^2}v_{\parallel}^i &= 0 \\
\frac{d^2}{d\lambda^2}v_{\perp}^i &= -\kappa v_{\perp}^i
\end{aligned} \tag{A3}$$

where  $v_{\parallel}^i = (\delta_{\beta}^i - J^i_{\beta})v^{\beta}$  and  $v_{\perp}^i = J^i_{\beta}v^{\beta}$ . The solutions for the given boundary conditions are elementary. Returning to the original variable  $\delta x^{\mu}$ ,

$$\begin{aligned}
\delta x^0(\lambda) &= \frac{\gamma(\lambda)}{\gamma(u)}\delta k^0(u)(\lambda - u) \\
\delta x_{\parallel}^i(\lambda) &= \frac{\gamma(\lambda)}{\gamma(u)}\delta k_{\parallel}^i(\lambda - u) \\
\delta x_{\perp}^i(\lambda) &= \frac{\gamma(\lambda)}{\gamma(u)}\delta k_{\perp}^i \sin_{\kappa}(\lambda - u)
\end{aligned} \tag{A4}$$

We can use the solution above to determine the angular diameter distance in the actual background spacetime. To do this, let  $k^{(0)\mu}(\lambda) = (1, -\gamma, 0, 0)$  and suppose an impulse at  $u$  given by  $\delta k^{\mu}(u) = \delta k_{\perp}^{\mu}(u) = (0, 0, -\gamma\epsilon, 0)$ , with  $\epsilon$  infinitesimal. Let  $\delta x^{\mu}(\lambda)$  be the solution to the Jacobi equation for this impulse,  $\delta x^{\mu}(\lambda) = \gamma(\lambda) \sin_{\kappa}(\lambda - u)(0, 0, \epsilon, 0) = (0, 0, r\epsilon, 0)$ . Since  $\delta x^{\mu}$  is a Jacobi vector,  $x^{(0)\mu}(\lambda) = (\lambda, r, 0, 0)$  and  $x^{(0)\mu}(\lambda) + \delta x^{\mu}(\lambda)$  are neighboring null geodesics to first order in  $\epsilon$ . The angle that their wavevectors make at  $u$  is  $\epsilon$ . The proper linear distance (in  $d\bar{s}^{(0)2}$ ) that they span at affine parameter  $\lambda$  is  $(\delta x^{\mu} \bar{g}_{\mu\nu}^{(0)} \delta x^{\nu})^{1/2}$ . The angular-diameter distance of our FRW background is defined as the ratio of this proper linear distance to the subtended angle,

$$\begin{aligned}
\bar{D}(u, \lambda) &= \frac{1}{\epsilon} \left( \delta x^{\mu} \bar{g}_{\mu\nu}^{(0)} \delta x^{\nu} \right)^{1/2} \\
&= a(\lambda) \sin_{\kappa}(u - \lambda)
\end{aligned} \tag{A5}$$

where the sense of the affine parameter in (A5) is that  $\lambda \leq u$ .



## Appendix B: Geodesic Deviation in Perturbed Spacetimes

Consider an arbitrary metric perturbed spacetime with metric

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} \quad (\text{B1})$$

with  $h_{\mu\nu}$  small. Let  $x_A^{(0)\mu}(\lambda)$  be an affinely parametrized geodesic of  $g_{\mu\nu}^{(0)}$ . Solutions,  $\zeta^{(0)\mu}(\lambda)$ , to the Jacobi equation of the background along  $x_A^{(0)\mu}(\lambda)$ ,

$$\frac{d^2 \zeta^{(0)\mu}}{d\lambda^2} + 2\Gamma_{\alpha\beta}^{(0)\mu} k^{(0)\alpha} \frac{d\zeta^{(0)\beta}}{d\lambda} + \Gamma_{\alpha\beta,\sigma}^{(0)\mu} k^{(0)\alpha} k^{(0)\beta} \zeta^{(0)\sigma} = 0 \quad (\text{B2})$$

generate nearby geodesics,  $x_B^{(0)\mu}(\lambda)$ , via  $x_B^{(0)\mu}(\lambda) = x_A^{(0)\mu}(\lambda) + \zeta^{(0)\mu}(\lambda)$ . Let  $x_A^\mu(\lambda)$  and  $x_B^\mu(\lambda)$  be geodesics of  $g_{\mu\nu}$  generated using the perturbative geodesic expansion from  $x_A^{(0)\mu}(\lambda)$  and  $x_B^{(0)\mu}(\lambda)$  respectively. We claim that, to first order,  $x_B^\mu(\lambda) - x_A^\mu(\lambda)$  solves the geodesic deviation equation of  $g_{\mu\nu}$  along  $x_A^\mu(\lambda)$ .

There is certainly nothing surprising in this claim. In fact, the assertion must be true if the PGE correctly generates nearby geodesics of  $g_{\mu\nu}$  as we claim it does. The explicit proof offered here can, thus, be thought of as another check on the method itself. Since the proof is nothing more than a tedious application of the usual perturbative techniques we present it only in schematic form.

To see that our assertion is true, we start with the Jacobi equation of  $g_{\mu\nu}$  along  $x_A^\mu(\lambda)$

$$\frac{d^2 w^\mu}{d\lambda^2} + 2\Gamma_{\alpha\beta}^\mu k_A^\alpha \frac{dw^\beta}{d\lambda} + \Gamma_{\alpha\beta,\sigma}^\mu k_A^\alpha k_A^\beta w^\sigma = 0 \quad (\text{B3})$$

By a Taylor expansion, the decomposition of the Christoffel terms into zeroth and first order expressions in the perturbation, the ansatz  $w^\mu = w^{(0)\mu} + w^{(1)\mu}$ , and the PGE identity  $k_A^\mu = k_A^{(0)\mu} + k_A^{(1)\mu}$ , we can express equation (B3) as two equations holding along  $x_A^{(0)\mu}(\lambda)$ , one containing only zeroth order terms and the other containing only first order terms. The zeroth order equation is seen to be equivalent to (B2), so that we already know that  $w^{(0)\mu}(\lambda) = x_B^{(0)\mu}(\lambda) - x_A^{(0)\mu}(\lambda)$  is a solution. The first order equation may be written  $T(w^{(1)}) = 0$  for some operator  $T$ . The specific form of  $T$  will not be necessary for our current purposes but it is not hard to obtain.

Consider now the equation obeyed by  $x_B^{(1)\mu}(\lambda) = x_B^\mu(\lambda) - x_B^{(0)\mu}(\lambda)$ ,

$$\frac{d^2 x_B^{(1)\mu}}{d\lambda^2} + 2\Gamma_{\alpha\beta}^{(0)\mu} k_B^{(0)\alpha} \frac{dx_B^{(1)\beta}}{d\lambda} + \Gamma_{\alpha\beta,\sigma}^{(0)\mu} k_B^{(0)\alpha} k_B^{(0)\beta} x_B^{(1)\sigma} = \Gamma_{\alpha\beta}^{(1)\mu} k_B^{(0)\alpha} k_B^{(0)\beta} \quad (\text{B4})$$

which holds along  $x_B^{(0)\mu}(\lambda)$  (PB). Using Taylor and  $k_B^{(0)\mu}(\lambda) = k_A^{(0)\mu}(\lambda) + \dot{w}^{(0)\mu}(\lambda)$ , where  $\cdot \equiv d/d\lambda$ , we can write (B4) as an equation along  $x_A^{(0)\mu}(\lambda)$ . If we subtract from this equation the equation for  $x_A^{(1)\mu}(\lambda)$ , which also holds along  $x_A^{(0)\mu}$  (and is identical to (B4) after the subscript  $B$ 's are replaced by  $A$ 's) we obtain the result  $T(x_B^{(1)} - x_A^{(1)}) = 0$  and our assertion is proved.

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**Figure 1.** Gravitational lensing by a single symmetric lens.  $\beta$  is the angle at the observer,  $o$ , between the image of the lens,  $l$ , and the unlensed image of the emitter,  $e$ .  $\theta$  is the angle between the image of the lens and the light ray  $epo$ , along which the emitter is observed. The point of deflection,  $p$ , lies in the lens plane. The intersection of the line joining the observer to the point of deflection with the source plane is labeled  $p'$ .  $\alpha$ , the deflection angle, is the angle at  $p$  between the image of the emitter,  $e$ , and  $p'$ .  $D(o, l)$  is the distance between the observer and the point of deflection,  $D(o, e)$  is the distance between the observer and the source plane,  $D(l, e)$  is the distance between the lens and source planes.

**Figure 2.** The lensing of a congruence of null geodesics. The actual congruence, with central ray  $x^\mu$ , joins the observer at  $o$  with the emitter at  $e$  and has area  $dA$  at the emitter. It is constructed from a congruence of the background, with central geodesic  $x^{(0)\mu}$ , reaching between the observer and a point  $q$ . The redshift in the background between the observer and  $q$  is equal to the redshift in the perturbed spacetime between the observer and the emitter. The background congruence has area  $dA^{(0)}$  at  $q$ .

**Figure 3.** The geometry of Section 4.  $\theta$  is the angle at the observer between the lens and the emitter.  $\lambda_l$  is the affine parameter value at which the background geodesic intersects the lens plane.  $\lambda_e$  is the affine parameter value at which the background geodesic intersects the source plane. The observer is located at affine parameter value  $\lambda_o$ .  $D_{ol}$  is the angular diameter distance in the background between the observer and the lens plane.